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Generalized Shooting Method for Analyzing Compliant Mechanisms

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Abstract **- We consider here a class of compliant mechanisms that consist of one or more flexible beams, the manipulation of which relies on the deflection of the flexible beams. As compared with traditional rigid-body mechanisms, compliant mechanisms have the advantages of no relative moving parts and thus involve no wear, backlash, noises and lubrication. In this paper, we present a formulation based on shooting method (SM) along with two numerical solvers to facilitate the analysis and the design process of a compliant mechanism. Unlike finite difference method (FDM) or finite element method (FEM) that offers accurate solutions at discrete nodes, the computed solution of SM, which treats the boundary value problem (BVP) as an initial value problem, is continuous. Three example compliant mechanisms are formulated to illustrate the generalized shooting method and the computed results are validated by comparing those obtained using FEM.**

Index Terms - Shooting method, compliant mechanism, Gauss-Newton method, flexible beam

I. INTRODUCTION

Compliant mechanisms have numerous applications in robotics/automation such as high precision manipulation [1], constant-force end effector [2], and micro-electro-mechanical-systems (MEMS). Additional applications of compliant mechanisms can be found in [3]. Unlike rigid-body mechanisms where actuations are applied at the joints connecting rigid members, manipulation of a compliant mechanism relies on deflection of the flexible beams. The compliant mechanisms, which have no relative moving parts (and thus involve no wear, backlash, noises and lubrication), are relatively compact in design and can be fabricated inexpensively. In order to fully exploit the advantages of compliant mechanisms for robotic/automation applications, we develop a numerical solver to facilitate the process of designing and prototyping a compliant mechanism, where classical small-deflection theory is not adequate for predicting the large deflection of an elastic member.

Four methods are commonly used to analyze a compliant mechanism; namely, elliptical integrals, FEM, chain algorithm, and the pseudo-rigid-body model. The geometrical solution to the 2nd order, nonlinear differential equation that characterizes the large deflection of flexible beams can be found in [4] but the derivation of the elliptical integrals is rather cumbersome and is useful for beams with relatively simple geometry, (see for example, [5]). *FEM* can deal with complicated geometric shape by discretizing structural members into small elements but the computation that depends on the resolution of the discretization is often time-consuming. Commercial FE software is widely available. However, its formulation is complicated. The *chain algorithm* [6] also discretizes the object being modeled into small linearized beam elements. Unlike FEM, the elements in the chain algorithm are analyzed in succession and hence, the inversion of overall stiffness matrix is avoided. Shooting methods are then used to satisfy boundary conditions. However, the accuracy of the results computed using the chain algorithm still depends on the resolution of the discretization. The *pseudo-rigid-body model* [3] finds the equivalent spring stiffness of a flexible beam by means of approximating functions. The beam is then decoupled into a torsional spring and a rigid link. Thus this method essentially extends the rigid-body analysis to approximate the end point deflection of a compliant mechanism with linear material properties; typical errors are within 0.5% of the closed-form elliptic integral solutions.

The fundamental member of a compliant mechanism is a flexible beam (where the axial dimension is much larger as compared to the dimensions in the cross-section). More recently, Yin *et al*. [7] evaluated three numerical methods for computing the deflected shape of a flexible beam against the exact closed-form solution by Frisch-Fay [4] for a uniform beam; shooting method (SM), finite difference method (FDM), and finite element method (FEM). The SM calculated shape (with 4th order Runge-Kutta or "ode45" in MATLAB) perfectly matches the Frisch-Fay solution with only a few iterations. The results demonstrate that the SM is superior to FDM and FEM (both in accuracy and computational time) in solving the large-deflection beam equation.While FDM or FEM offers accurate solutions at discrete nodes, the SM computed solution, which treats the boundary value problem (BVP) as an initial value problem, is continuous. As compared with the four existing methods discussed earlier, the formulation of SM is simple and efficient because it does not rely on discretization of compliant links. It can also deal with nonlinear material properties. This provides the motivation to develop a generalized SM for analyzing compliant mechanisms.

The shooting method was originally suggested by Keller [8]. By guessing the unknown initial values and then integrating the ODE's, the SM *shoots* at the terminal values iteratively until convergence. Like solving nonlinear equations, two major concerns of SM are the need for the reasonably close guesses and the convergence of solutions. Multiple-SM has been developed by Keller [8] and Stoer *et*

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al. [9] to overcome convergence problems, which divides the interval of independent variable into small subdivisions and then performs SM on each of the subdivisions such that after pieced together, the solution is continuous. Another technique is modified SM [10] which shoots at intermediate values in succession until terminal value. A major application of SM has been the analysis of structural members. Wang *et al.* [11] analyzes rectangular frames and circular rings using shooting-optimization techniques. Pai *et al.* [12] use multiple-SM to solve the problem of flexible beams undergoing large 3D deflections. Most of these research efforts focused on the use of SM for solving the deflection shape of a beam. Its use for analyzing compliant mechanisms remains under-exploited.

The remainder of this paper offers the following:

1. We generalize the SM for solving two-point BVP with an application to analyze complaint mechanisms.
2. Two numerical algorithms are given to implement generalized SM, namely, unconstrained Gauss-Newton method and constrained Gauss-Newton method.
3. We illustrate with three examples the formulation of the generalized SM for a broad spectrum of applications.
4. We validate our numerical method by comparing the computed results against those obtained with FEM.

II. FORMULATION OF GENERALIZED SM

Consider a system of ℓ uncoupled, normalized sets of 1st order nonlinear ordinary differential equations (ODE's):

$$\begin{aligned} \mathbf{q}_1' &= \mathbf{f}_1(u_1, \mathbf{q}_1, \xi_1) \\ &\vdots \\ \mathbf{q}_\ell' &= \mathbf{f}_\ell(u_\ell, \mathbf{q}_\ell, \xi_\ell) \end{aligned} \tag{1}$$

where $\mathbf{q}_i' = d\mathbf{q}_i / du_i$;

$0 \le u_1, \cdots, u_\ell \le 1$ are independent variables;

$\mathbf{q}_i = [q_{i1} \quad q_{i2} \quad \cdots]^{\mathrm{T}}$ is the state vector, $i = 1, \ldots, \ell$; and

$\xi = [\xi_1^T, \xi_2^T, \cdots, \xi_\ell^T]^T$ is a vector of *r* unknown parameters needed in order to integrate the ODE.

Equation (1) is subject to the initial and terminal values (BV's) denoted as

$$\mathbf{q}(\mathbf{u}=0) = \boldsymbol{\mu} \text{ and } \mathbf{q}(\mathbf{u}=1) = \boldsymbol{\eta}$$

where $\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_\ell^T]^T$ is a $n\times 1$ vector. We consider here a class of problems where only some of the initial values are known. It is worth noting that *the original formulation in the conventional SM does not involve unknown parameters* and thus, can only deal with a special case of (1) where all the parameters (*ξ's)* are known.

The shooting method transforms a boundary value problem (BVP) to an initial-value-problem (IVP). It generally requires making guesses on the unknown initial BV's in order for the nonlinear ODE's to be solved. Since ξ is an unknown vector, additional guesses on the values of the parameters are also needed. These guesses must satisfy two sets of constraints after solving the ODE's; namely, the terminal constraints and the physical constraints. We denote the *m* unknown initial values of $\mathbf{q}(u=0)$ by an $m\times 1$ unknown vector $\boldsymbol{\mu}_u$. In order for this class of problems to be solvable, these constraints must together form *r+m* nonlinear algebraic equations. They are expressed mathematically in (2a) and (2b):

Constraint Set I: Satisfying (known) terminal constraints

$$g_i(\boldsymbol{\mu}_u, \xi) = \sum_{j=1}^{n} c_{ij}\, \eta_j^*(\boldsymbol{\mu}_u, \xi) + d_i = 0, \quad i = 1, 2, \ldots, p \tag{2a}$$

where the function $\eta_j^*(\cdot)$ calculates the j^{th} terminal values η_j for any given $\boldsymbol{\mu}_u$ and ξ; and c_{ij} and d_i are constants.

Constraint Set II: Satisfying (known) physical constraints

$$g_i(\boldsymbol{\mu}_u, \xi) = 0, \quad i = p+1, p+2, \ldots, r+m \tag{2b}$$

We assume the solution of the BVP represented by (1) and (2) exists and is unique. For the existence and uniqueness theorem of BVP, please refer to Stoer *et al.* [9].

A general system of nonlinear algebraic equations (NAE) requires that the number of initial guesses equals the number of NE. The BVP, Equations (1) and (2), share this property and can be treated as implicit NAE. Numerical solvers for NAE can thus be used to solve the BVP. However, unlike NAE of a typical IVP, the exact derivatives of the BVP are unavailable and must be approximated numerically. We choose to use Gauss-Newton method that requires only the 1st derivative, as opposed to Newton's method that requires both the 1st and 2nd derivatives. The following section discusses two Gauss-Newton based solvers to solve the BVP.

The Gauss-Newton method begins with a set of initial guesses $\mathbf{x}_0$ for the unknown initial values and parameters, and solves for $\mathbf{x}$ iteratively. In any (say k^{th}) iteration, we approximate the constraint vector $\mathbf{g}(\boldsymbol{\mu}_u, \xi)$ at iteration in terms of the guesses $\mathbf{x}_k$ by the 1st-order function:

$$\mathbf{g}(\mathbf{x}) \cong \mathbf{m}(\mathbf{x}) = \mathbf{g}(\mathbf{x}_k) + \mathbf{J}(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) \tag{3}$$

where $\mathbf{x} \in R^{r+m}$ is a vector of the guessed initial values and parameters, namely $\mathbf{x} = [\boldsymbol{\mu}_u^T \quad \xi^T]^T$;

$\mathbf{J}(\mathbf{x}_k)$ is the Jacobian matrix evaluated numerically (for example, finite-difference-method) at $\mathbf{x}_k$; and

$\mathbf{g}(\mathbf{x}) \in R^{r+m}$ is the constraint vector given by (2a) and (2b). The vector function $\mathbf{g}(\mathbf{x})$ should be zero if the problem is solved.

The remaining problem is then to find a vector $\mathbf{x}$ that minimizes the approximation function:

$$\min \sum_{i=1}^{r+m} \left[m_i(\mathbf{x})\right]^2 \tag{4}$$

In the following two subsections, we consider two different cases; namely, with and without bounds on the unknowns. Case I (unbounded) is solved as a least square (L-S) problem. Case II is formulated as a quadratic programming (QP) problem, which allows the bounds to be imposed on the unknowns. Computational steps will be given for each case.

II.1 Unbounded Gauss-Newton Method:

No bound imposed on the unknowns –solved as a L-S problem

Equation (4) is essentially a linear least-square (L-S) problem because the approximation functions can be expressed as

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$$\sum_{i=1}^{r+m}[m_i(\mathbf{x})]^2 = \|\mathbf{J}_k\mathbf{x} - \mathbf{b}_k\|^2 \quad (5)$$

where $\mathbf{J}_k = \mathbf{J}(\mathbf{x}_k)$ and $\mathbf{g}_k = \mathbf{g}(\mathbf{x}_k)$ are defined to simplify the notation; and

$$\mathbf{b}_k = \mathbf{J}(\mathbf{x}_k)\mathbf{x}_k - \mathbf{g}(\mathbf{x}_k) = \mathbf{J}_k\mathbf{x}_k - \mathbf{g}_k \quad (6)$$

The L-S problem has a formal solution via matrix algebra

$$\mathbf{x} = \mathbf{x}_{k+1} = (\mathbf{J}_k^T\mathbf{J}_k)^{-1}\mathbf{J}_k^T\mathbf{b}_k \quad (7)$$

which replaces $\mathbf{x}_k$ for the next iteration until

$$\|\mathbf{g}(\mathbf{x})\| \le tolerence\,.$$

Using (6), (7) can be written as

$$\mathbf{x}_{k+1} = (\mathbf{J}_k^T\mathbf{J}_k)^{-1}\mathbf{J}_k^T(\mathbf{J}_k\mathbf{x}_k - \mathbf{g}_k) = \mathbf{x}_k - (\mathbf{J}_k^T\mathbf{J}_k)^{-1}\mathbf{J}_k^T\mathbf{g}_k$$

It is assumed here that $(\mathbf{J}_k^T\mathbf{J}_k)^{-1}$ exists. This is a line search procedure along the descent direction $\mathbf{v}$ defined as

$$\mathbf{v} = -(\mathbf{J}_k^T\mathbf{J}_k)^{-1}\mathbf{J}_k^T\mathbf{g}_k \quad (8)$$

In order to know how far $\mathbf{x}_k$ should go along the descent direction, a positive step-length factor β is introduced here

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \beta\mathbf{v} \quad (9)$$

such that $\mathbf{g}(\mathbf{x}_k + \beta\mathbf{v})$ is minimum. In order to obtain the value of β, we choose three numbers $\beta_1<\beta_2<\beta_3$ that are close to β. We then construct a quadratic interpolation polynomial $p(z)$ from the three numbers. The minimum of $p(z)$ can be obtained at

$$\hat{\beta} \in [\beta_1, \beta_3]\,,$$

which approximates the minimum of $\mathbf{g}(\mathbf{x}_{k+1})$. The steps for the unbounded Gauss-Newton method are outlined below:

Computational steps for Case I:

Given initial $\mathbf{x}_0$, tolerance ε, and $(\beta_1, \beta_2, \beta_3)$, repeat the following steps until ε is met.

1. Evaluate $\mathbf{v} = -(\mathbf{J}_k^T\mathbf{J}_k)^{-1}\mathbf{J}_k^T\mathbf{g}_k$.
2. Calculate the optimal $\hat{\beta}$ from $p(z)$.
3. If $\|\mathbf{g}(\mathbf{x}_k + \hat{\beta}\mathbf{v})\| \le \varepsilon$ return $\mathbf{x}$
   else $\mathbf{x}_{k+1} = \mathbf{x}_k + \hat{\beta}\mathbf{v}$
   End

II.2 Bounded Gauss-Newton Method:

Bounded unknowns– solved as a quadratic programming (QP) problem

The SM for solving BVP is faster than other numerical methods such as FDM or FEM. The drawback of SM, however, is that initial guesses are needed to shoot for the specified terminal BV's on the other side of the domain. Bad initial guesses could lead to wrong answer or even divergence. Most practical physical problems, however, have bounded BV's and often these bounds are one-sided, positive or negative. For this reason, we offer an alternative formulation using knowledge of the problem to impose a bound on the unknowns in solving (1), (2a) and (2b). It is hope that any feasible initial guesses will lead to the true solution.

Expanding the approximation function in (5),

$$\min\|\mathbf{J}_k\mathbf{x} - \mathbf{b}_k\|^2 = \min\left(\mathbf{x}^T\mathbf{J}_k^T\mathbf{J}_k\mathbf{x} - 2\mathbf{b}_k^T\mathbf{J}_k\mathbf{x} + \mathbf{b}_k^T\mathbf{b}_k\right) \quad (10)$$

Since $\mathbf{b}_k^T\mathbf{b}_k$ is a constant in every k^{th} step, it can be dropped from the minimization problem. The least square problem can be recast as a QP problem

$$\min\left(\mathbf{x}^T\mathbf{J}_k^T\mathbf{J}_k\mathbf{x} - 2\mathbf{b}_k^T\mathbf{J}_k\mathbf{x}\right),\quad \mathbf{x}_{lb} \le \mathbf{x} \le \mathbf{x}_{ub} \quad (11)$$

where $\mathbf{x}_{lb}$ and $\mathbf{x}_{ub}$ defines the lower and upper bounds of the initial guesses respectively. Equation (11) is a standard QP problem that can be solved using several different methods [13]. After solving the QP, the optimal solution will be the next step $\mathbf{x}_{k+1}$. The descent direction $\mathbf{v}$ can be calculated as

$$\mathbf{v} = \mathbf{x}_{k+1} - \mathbf{x}_k \quad (12)$$

The procedure for finding β that minimize $\mathbf{g}(\mathbf{x}_{k+1})$ is the same as the previous section.

Computational steps for Case II:

Given initial $\mathbf{x}_0$ such that $\mathbf{x}_{lb} \le \mathbf{x}_0 \le \mathbf{x}_{ub}$, tolerance ε, and, $(\beta_1, \beta_2, \beta_3)$ repeat the following steps until ε is met

1. For every k^{th} step, solve the QP or (11) for optimal $\mathbf{x}_{k+1}$
2. Evaluate $\mathbf{v} = \mathbf{x}_{k+1} - \mathbf{x}_k$.
3. Calculate the optimal $\hat{\beta}$ from $p(z)$.
4. If $\|\mathbf{g}(\mathbf{x}_k + \hat{\beta}\mathbf{v})\| \le \varepsilon$ return $\mathbf{x}$
   else $\mathbf{x}_{k+1} = \mathbf{x}_k + \hat{\beta}\mathbf{v}$
   End

The formulation and solution method discussed in Case II offers two advantages:

(a) The QP avoids inverting $\mathbf{J}_k^T\mathbf{J}_k$, which may be singular.

(b) Some knowledge on the bounds of the unknown BV parameters ensures that the solution is always in the feasible region.

While developed for the shooting method, it is worth noting that this bounded Gauss-Newton method can be used to solve general nonlinear equations. As long as we know the upper bounds and lower bounds of the true roots, we can apply bounded Gauss-Newton method to avoid inverting $\mathbf{J}_k^{\mathrm{T}}\mathbf{J}_k$.

III. APPLICATIONS TO COMPLIANT MECHANISMS

A general compliant mechanism includes multiple compliant links connected by pinned or clamped joints, the formulation of which can be represented in the form suggested by (1). We illustrate here a systematic approach leading to a set of guidelines for formulating a compliant mechanism. The following begins with the governing equation for a flexible beam capable of large deflection. This is followed by formulating the flexible beam as a member of a compliant mechanism, along with the method for identifying the unknowns and the complete set of constraint equations.

III.1. Governing Equation of a Flexible Beam

Fig. 1 shows a flexible beam of length L deflected under a point force F along the direction α at the location C.

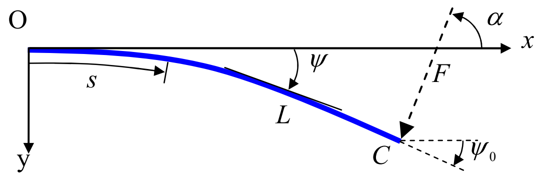

Fig. 1 Schematics of a typical flexible beam

In Fig. 1, the coordinate frame O is attached at one end of the beam, where the x axis lies on the un-deflected beam

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that deflects in the y direction. The location of C is denoted as $\mathbf{P}_C = [x_C \quad y_C]^{\mathrm{T}}$.

The differential equation of a flexible link is given by

$$EI(s)\frac{d\psi}{ds} = M = F\sin\alpha(x - x_c) + F\cos\alpha(y - y_c) \qquad (13)$$

where s is the arc length of the beam as shown in Fig. 1;

E is the Young's module of the beam material;

I(s) is the moment of area of the beam;

ψ is the angle of rotation;

M is the bending moment (positive when it produces compression in the lower part of the beam); and

F is the applied force (positive when pointing towards the positive y-direction.)

In order to express (13) explicitly in terms of s, we differentiate (13) with respect to s leading to the following 2nd order differential equation:

$$EI(s)\frac{d^2\psi}{ds^2} + E\frac{dI(s)}{ds}\frac{d\psi}{ds} = -F\sin\alpha\cos\psi - F\cos\alpha\sin\psi \qquad (14)$$

where $\cos\psi = {dx}/{ds}$ and $\sin\psi = {dy}/{ds}$. Furthermore, we normalize (14) leading to a dimensionless nonlinear differential equation of θ with respect to u,

$$I(u)\frac{1}{L^2}\frac{d^2\theta}{du^2} + \frac{dI(u)}{du}\frac{1}{L^2}\frac{d\theta}{du} + \frac{F}{E}\sin\theta = 0 \qquad (15)$$

where $\quad u = {s}/{L} \in [0,1] \qquad (15a)$

and $\quad \theta = \psi + \alpha \in [\alpha, \alpha + \psi_0] \qquad (15b)$

Since the force F and its direction α may not be known in advance, they are treated as unknown parameters.

Once (15) is solved, the position C can be obtained by the following equation:

$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = L\int_0^1 \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} du = L\int_0^1 \begin{bmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{bmatrix} du \qquad (16)$$

III.2 Beam as a Member in a Compliant Mechanism

Consider the flexible beam in Fig. 1 as the i^{th} link of a compliant mechanism. Equation (15) that governs the large deflection of the link can be written in the form as (1) by defining $\theta_{i1} = \theta$ and $\theta_{i2} = d\theta / du = \theta_{i1}'$:

$$\boldsymbol{\theta}_i' = \mathbf{f}_i(u_i, \boldsymbol{\theta}_i, \boldsymbol{\lambda}_i) \qquad (17)$$

where $\quad \mathbf{f}_i = [f_{i1} \quad f_{i2}]^{\mathrm{T}}$,

$$f_{i1} = \theta_{i2}$$

$$f_{i2} = -\frac{I_i'(u_i)}{I_i(u_i)}\theta_{i2} - \frac{F_i L_i^2}{E_i I_i(u_i)}\sin\theta_{i1}$$

The initial values needed in order to solve (17) are $\theta_{i1}(0)$ and $\theta_{i2}(0)$. For a general compliant link as shown in Fig. 1, the initial values depend on the type of joint at O:

Conditions at O	**Initial Values**	
	Known	**Unknown**
Clamped	$\theta_{i1}(0) = \alpha_i$	$\theta_{i2}(0)$
Pinned	$\theta_{i2}(0) = 0$	$\theta_{i1}(0)$

Hence, the i^{th} compliant link has m=1 unknown initial value and r=2 unknown parameters ($\boldsymbol{\lambda}_i = [F_i \quad \alpha_i]^T$). Similarly, a compliant mechanism composed of k compliant links would require $3k$ guesses since each compliant link has one unknown initial value and two unknown parameters.

III.3 Formulation of Constraint Equations

The constraint sets in (2a) and (2b) for a compliant mechanism can be found at C that connects to another link by a pinned or clamped joint, which may be rigid or compliant as shown in Fig. 2.

Constraint Set I:

For a pinned joint connection at C, the constraint equations that correspond to (2a) are

$$\theta_{i2}(1) = 0 \quad i = 1,2 \qquad (18)$$

For a clamped joint at C, the constraints have the following form:

$$c_{11}\theta_{11}(1) + c_{13}\theta_{21}(1) + d_1 = 0 \qquad (19a)$$

$$c_{22}\theta_{12}(1) + c_{24}\theta_{22}(1) + d_2 = 0 \qquad (19b)$$

Equation (19a) states that link 1 and 2 form a constant clamped angle at C while (19b) is a moment balance equation at C.

Constraint Set II:

The physical constraints are derived from the free body diagram as shown in Fig. 2(b) with Newton's 3rd law:

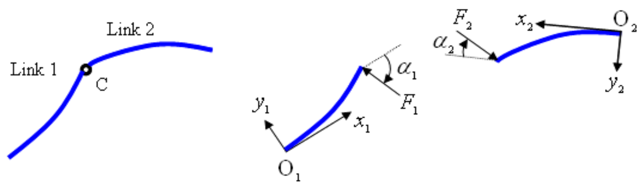
$$F_1 - F_2 = 0 \qquad (20a)$$

$$\alpha_1 - \alpha_2 - \phi = 0 \qquad (20b)$$

where ϕ is the relative orientation between frame 1 and frame 2. Since C is a common point for both links, its locations in both frames are the same after transformation

$$\mathbf{R}_2\mathbf{P}_C + {}_2\mathbf{P}_1 = {}_1\mathbf{P}_C \qquad (20c)$$

where $\mathbf{R}$ is the transformation matrix from frame 2 to frame 1; ${}_i\mathbf{P}_C$ is the position vector of C in frame, i=1,2; and ${}_2\mathbf{P}_1$ is the position vector of O_2 expressed in frame 1.



(a) Schematics (b) Free body diagram

Fig. 2 A pair of connecting links

From the above formulation we can see that the basic procedures for solving CM problems are (1) identifying unknown initial values and parameters needed for solving ODE of compliant links (2) identifying constraint equations from joints and rigid segments (3) match the number of unknowns and the number of constraint equations, then solve with Gauss-Newton Method.

IV. NUMERICAL EXAMPLES

In order to illustrate the formulation in Section III for a broad spectrum of applications, we consider three examples shown in Fig. 3(a), 3(b) and 3(c). Example I shows a pair of compliant links directly connected by a pin joint. Unlike Example I where the linear motion δX is controlled by the link deflection as shown in Fig. 3(a), the two flexible links in Example II are clamped to a rigid

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member and the input displacement is applied at the joint as shown in Fig. 3(b). In Example II, the constraint equations come from the rigid segment. A special case, where both links are clamped to a fixed frame is commonly used for precision manipulation of camera lens. Example III is a special class of compliant mechanism where the compliant link is in contact with a smooth object. Unlike other compliant mechanisms discussed above, the contact problem involves no joints. In addition, the point where the force applies is not known in advance for a contact problem and hence, the length from origin to contact point L must be treated as an unknown. Table 1 lists the unknown parameters/initial values for the three examples.

Table 1 Summary of parameters in Examples I, II and III

Ex.	ℓ	r	ξ	m	μ_u	p
I	2	4	$F_1, \alpha_1, F_2, \alpha_2$	2	$\theta_{12}(0), \theta_{22}(0)$	2
II	2	5	$F_1, \alpha_1, F_2, \alpha_2, P$	2		2
III	1	3	F, α, L	1	$\theta_2(0)$	1

The constraint sets I and II (which correspond to terminal and physical constraints respectively), and selected numerical results for each example are given as follows:

Example 1: Compliant Slider Mechanism (CSM)

We expect $r+m$=6 constraint equations; namely, p = 2 terminal and $r+m-p$=4 physical constraint equations:

Constraint Set I: The two equations are given by (18) for a pinned joint.

Constraint Set II: They are written in similar forms as (21):

$$F_1 - F_2 = 0 \tag{21a}$$

$$\alpha_1 - \alpha_2 - \pi/2 + \varphi = 0 \tag{21b}$$

$$\begin{bmatrix} \cos(\pi-\phi) & -\sin(\pi-\phi) \\ \sin(\pi-\phi) & \cos(\pi-\phi) \end{bmatrix} \begin{bmatrix} x_{2C} \\ y_{2C} \end{bmatrix} + \begin{bmatrix} \delta X \\ 0 \end{bmatrix} = \begin{bmatrix} x_{1C} \\ y_{1C} \end{bmatrix} \tag{21c}$$

Hence, the numerical solver in Section II can be used to solve for the six (r+m) unknowns and the deflected shape of the links from the two ODE's in Equation (1) and the six constraint equations given by the constraint sets. Fig. (4) shows the result of varying δX, where the displacement of the slider along the x_1 direction is chosen as input.

Example 2: Compliant Parallel Mechanism (CPM)

We expect to need $r + m$=7 constraint equations, which are from the rigid member connecting the two flexible links. The constraint equations are similar to (18)~(20). They are obtained as follows.

Constraint Set I (p = 2): Two terminal constraints must be satisfied after solving (1):

$$g_i(\mu_u, \xi) = [\theta_{i1}(1) - \alpha_i] - \phi = 0 \quad i = 1,2 \tag{22}$$

where $\phi = \tan^{-1} \dfrac{x_A - x_B}{y_B + L_{AB} - y_A}$; L_{AB} is the distance between points A and B; [x_A y_A] and [x_B y_B] are expressed in frame 1 and 2 respectively.

Constraint Set II (r+m-p=5): The equations are as follows. The deflected position of link A is the same as the input.

$$g_3 = y_A - \delta X = 0 \tag{23a}$$

Since the member L_{AB} is rigid, we have

$$g_4 = (x_A - x_B)^2 + (y_B + L_{AB} - y_A)^2 - L^2 = 0 \tag{23b}$$

Summing the forces applied to link 1 at point A leads to

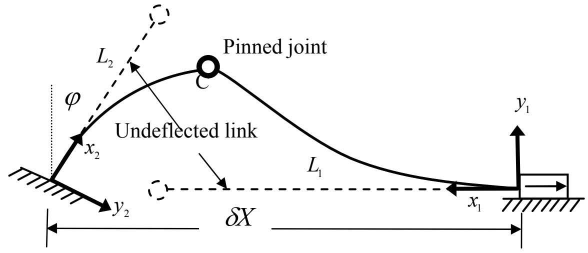
$$F_1 = [(P - F_2 \sin\alpha_2)^2 + (F_2 \cos\alpha_2)^2]^{1/2} \tag{23c}$$

$$\alpha_1 = \pi/2 - \tan^{-1}\left(\frac{P - F_2 \sin\alpha_2}{F_2 \cos\alpha_2}\right) \tag{23d}$$

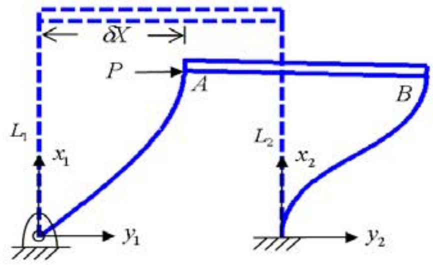
Balancing the moment of forces on the rigid link, we have

$$\frac{EI}{L_1}\theta_{12}(1) + \frac{EI}{L_2}\theta_{22}(1) + F_2 L_{AB} \cos(\alpha_2 + \phi) = 0 \tag{23e}$$

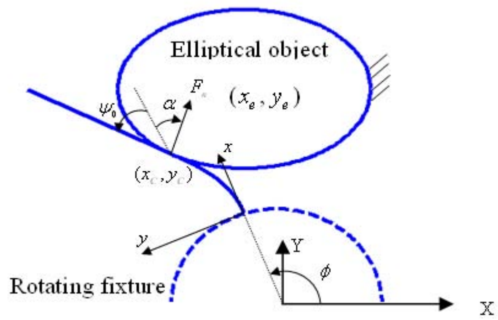
Fig. 5 shows the loci of points A and B of the rigid link and the force required at A, against results computed using FEM. As the input displacement δX increases, the rigid link would tilt (A is higher than B) and is not horizontal. It is worth noting that the method of pseudo-rigid-body model is unable to predict the rotation of the rigid link.



(a) Example I: Compliant Slider Mechanism (CSM)



(b) Example II: Compliant Parallel Mechanism (CPM)



(c) Example III: Compliant Contact Mechanism (CCM)

Fig. 3 Illustrative examples

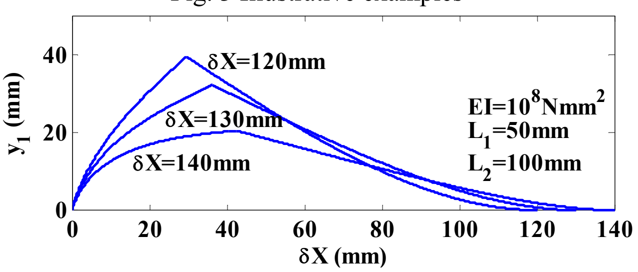


Fig. 4 Numerical results for Example I

Example 3: Compliant Contact Mechanism (CCM)

We need r+m=4 constraint equations at the contact point C. We consider here a rotating link in contact with a moving, frictionless elliptical object.

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Constraint Set I (*p=1*): The terminal constraint equation is

$$g_1(\boldsymbol{\mu}_u,\xi)=\theta_2(1)=0 \tag{24}$$

Constraint Set II (*r+m-p=3*): Three physical constraints imposed at (x_C, y_C) lie on the object peripheral, which can be obtained by (16). The equations are the equation of an ellipse described in the link frame:

$$g_2(\boldsymbol{\mu}_u,\xi)=f_e(x_C,y_C)=0 \tag{25a}$$

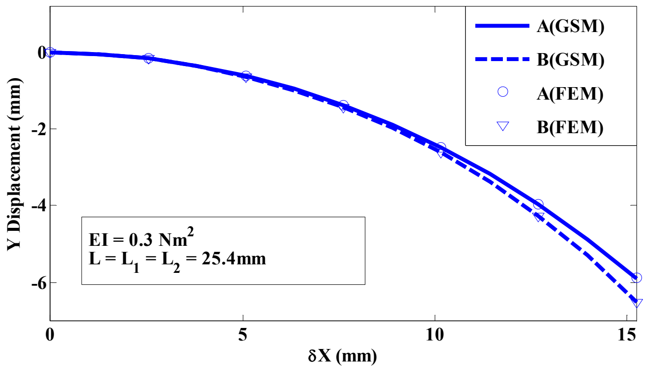
Also, the link and the object share the same slope at the contact point, and since $\psi_0=\theta_1(1)-\alpha$. We have

$$g_3(\boldsymbol{\mu}_u,\xi)=\left.\frac{\partial f_e/\partial x}{\partial f_e/\partial y}\right|_{(x=x_C,y=y_C)}+\tan\psi_0=0 \tag{25b}$$

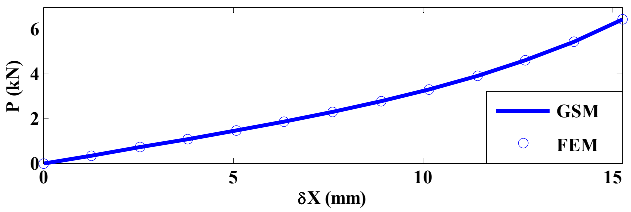
Since the contact is frictionless, the contact force F_n will be normal to the link at C.

$$g_4(\boldsymbol{\mu}_u,\xi)=\theta(1)-\pi/2=\alpha+\psi_0-\pi/2=0 \tag{25c}$$

Fig. 6 compares the result against those computed using FEM. The angular position of the rotating fixture is $\phi=-0.1524x_e+102°$, where ϕ is in degrees and x_e is the center of the ellipse in meter.



(a) Vertical displacement of point A and B



(b) Force required at point A

Fig. 5 Numerical results for Example II

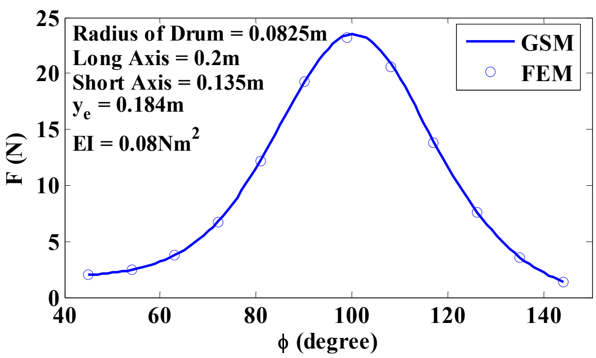


Fig. 6 Numerical results for Example III

V. CONCLUSIONS

A general formulation along with two numerical solvers (Gauss-Newton and constrained Gauss-Newton methods) has been presented for analyzing compliant mechanisms where the manipulation relies on the deflection of the flexible beams.

Three examples were given to illustrate the applications of the formulation for analyzing compliant mechanisms. The numerical solutions of these examples closely agree with those computed using FEM. As illustrated, the generalized SM (which treats the BVP as an IVP) offers several advantages as compared to its counterpart. These advantages include the following: (a) Its formulation is simple; (b) the computation is efficient because it does not rely on discretization of compliant links; and (c) its solution is continuous.

ACKNOWLEDGEMENT

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